CERTAINTY IN MATHEMATICS: IS THERE A PROBLEM?

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**Abstract**

Two independent questions about the certainty in mathematics are posed. Is mathematical knowledge known with certainty? Why is the belief in the certainty of mathematical knowledge so widespread and where does it come from? Absolutist claims of the certainty of mathematical knowledge are articulated and critiqued. The contrasting view that mathematical knowledge is known with a certainty circumscribed by the limits of human knowing is proposed, elaborated and defended. In explaining the reasons for beliefs in the certainty of mathematics both cultural historical and individual psychological factors are identified. The cultural historical development of mathematics contributes four factors: 1, the invariance and conservation of number and the reliability of calculation; 2, the emergence of numbers as abstract entities with apparently independent existence; 3, the emergence of proof with its goal of convincing readers of the certainty of mathematical propositions; 4, the engulfment and neutralisation of historically emergent contradictions and uncertainties and their incorporation into the mathematical narrative of certainty. The second source of the beliefs in the certainty of mathematics is the individual development of learners who internalize ideas of invariance, reliability and certainty through their classroom experiences and exposure to cultural factors including these four.

**Introduction**

One of the defining characteristics of mathematics is its claims to providing knowledge that is known with certainty. In this paper I explore both the basis for these claims of certainty and the basis for belief in these claims. But I should first mention that mathematics also produces uncertainties. In addition to the speculative uncertainty when a learner or mathematician formulates a conjecture, there is also the domain of doubts and uncertainties that can be experienced by learners when confronted by the absolute monolith of mathematics. Certainty in mathematics immediately raises thoughts about its opposite, and the role of uncertainty in learning and doing mathematics. I will address the issue of how mathematics deals with new uncertainties that arise in its historical development. But in the limited space here all I can do is to acknowledge that there are also personal questions of uncertainty, and then put them to one side while focussing on the questions of certainty.

Two problems of certainty in mathematics can be formulated. The first concerns mathematical knowledge. It is widely claimed that mathematical knowledge is absolutely certain. But does this claim stand up? Is mathematical knowledge objectively certain, beyond any doubt?

The second problem is related but distinct. Many believe that mathematical knowledge is known with certainty. What is the basis for such beliefs? Where do these beliefs come from and how are they formed?

It is important to distinguish these two problems. The fact that mathematical knowledge is true with certainty does not, of itself, cause belief in its certainty. Certainty of itself does not explain why there is widespread belief in certainty, or where such belief originates from. Something can be true and one can still fail or refuse to believe it, or even believe it for the wrong reasons. Just as a mathematical proposition needs a proof to warrant it as objective mathematical knowledge, so too a mathematical claim needs to have some underpinning reason or basis for us to believe it subjectively. So what makes us believe in the truth of mathematical theorems, from statements like '2+2=4', the Pythagorean theorem, through to more advanced mathematical results? My answer to this question is that there are three types of reasons. First, there are foundational and logical reasons, that is convincing demonstrations of truth. Second, there are cultural and historical reasons which include the evolution of mathematical concepts and knowledge in specific historical ways to serve particular cultural and social needs, including the reliability and certainty of mathematical knowledge. Third, there are personal cognitive reasons for belief based on our induction into mathematical practices and methods. These shape our expectations and views of mathematics, including the development and reinforcement of beliefs in the certainty of mathematical knowledge.[[1]](#footnote-1)

For example, consider the simple truth '2+2=4'. There are foundational and logical justifications for this truth, and a simple formal proof establishes it on the basis of definitions and the axioms of arithmetic. However, most people believe this statement without having ever seen such a proof. There are cultural and historical developments that have provided the conceptual and symbolic underpinnings of this truth. The ways that the constituent concepts of '2', '4', '+' and '=' are formulated and conceptualised virtually guarantee a belief in the certainty of this truth over time. However, persons need exposure to, and assimilation of, such conceptualisations and reasoning before they believe in the truth of the statement. Thus there is substantial personal learning and cognitive development that needs to be undergone by learners to come to understand and believe in the truth of mathematical facts like '2+2=4', as well as more complex mathematical results.

Why are the problems of certainty relevant to mathematics education? The arguments about certainty that follow draw on the philosophy and foundations of mathematics, the history and anthropology of mathematics, and the psychology of learning mathematics. None of these fields of itself welcomes interdisciplinarity in the way that contemporary research in mathematics education does. So this is a good reason as to why these issues are discussed within mathematics education. However, there is a deeper reason. One area of contemporary theory in mathematics education research, the philosophy of mathematics education, has long argued that philosophical conceptions of mathematics have a profound impact on its teaching and learning (Ernest, 1991). Being aware of how a belief in the certainty of mathematics is constructed, both historically and individually, has important consequences for planning and theorizing the teaching of mathematics. Furthermore, being aware of the limits of the widespread claims of the certainty and the objectivity of mathematics is also important within mathematics education, both for theoretical reasons and because of the implications for both mathematics teaching and mathematics teacher education.

**Certainty and Objectivity**

Certainty in its original meaning is a mode of belief, the strongest mode of positive belief and epistemological commitment. Beliefs held with certainty are those to which their holders admit no doubts and which they understand can withstand any challenges and sceptical questioning, no matter how strong. Such beliefs are regarded as infallible and irrefutable within current epistemological frameworks. However, the idea of certainty has been expanded beyond an attitude held by persons towards beliefs. Certainty can also be attributed to the objects of knowledge themselves, the propositions expressing beliefs. When there is strong agreement on them, these can also be described as certain, or possessing certainty, provided that they are viewed as objectively warranted and currently able to withstand all doubts, questioning or challenges to their veracity that persons might level at them.

However, there is an outstanding controversy in mathematics and its philosophy concerning the certainty of mathematical knowledge and what it means. The traditional absolutist view is that mathematics provides infallible certainty that is both objective and universal. According to this view, mathematical knowledge is absolutely and eternally true and infallible, independent of humanity, at all times and places in all possible universes. When correctly formulated mathematical knowledge is forever beyond error and correction. Any possible errors in published results, should they occur, are down to human error, comprising carelessness, oversight or mis-formulation. From this perspective certainty, objectivity and universality are essential defining attributes of mathematics and mathematical knowledge.

In contrast, there is an alternative ‘maverick’ tradition in the philosophy of mathematics according to which mathematical knowledge is humanly constructed and fallible (Kitcher and Aspray, 1988). This tradition includes the perspectives known as fallibilism (Lakatos 1976), humanism (Hersh 1997) and social constructivism (Ernest 1998). As one of the key founders of this tradition puts it:

Why not honestly admit mathematical fallibility, and try to defend the dignity of fallible knowledge from cynical scepticism, rather than delude ourselves that we shall be able to mend invisibly the latest tear in the fabric of our ‘ultimate’ intuitions. (Lakatos 1962: 184)

In this quotation Lakatos is reacting against the claim (and hope) that absolute foundations for mathematical knowledge can be found, the vain hope aspired to by foundational movements and schools in the philosophy of mathematics in the early part of the twentieth century.

The maverick tradition rejects the claim of the absolute and universal truth of mathematical knowledge (Ernest 1991, 1998, Hersh 1997, Tymoczko 1986). It argues that for a number of reasons mathematical knowledge does not constitute objective truth that is valid for all possible knowers and all possible places and times. In this tradition, while the certainty of mathematical knowledge is still acknowledged, the concept of certainty is circumscribed and limited to current human knowing.

This ambiguity in the use of the term certainty is best understood in terms of the concept of objectivity. On the one hand, what I term absolute objectivity refers to knowledge that is validated in the physical world as a brute fact verifiable by the senses, or in the domain of non-empirical knowledge by dint of logical necessity. On the other hand, what I term cultural objectivity refers to knowledge that has a warrant that goes beyond any individual knower’s beliefs, thus it is the opposite of subjectivity. Laws, money and language are culturally objective because their existence is independent of any particular person or small groups of persons, but not of humankind as a whole. These two meanings are evidently not the same because mathematical objects could consistently exist in the social and cultural realm beyond any individual beliefs (cultural objectivity) without having independent physical existence or existence due to logical necessity (absolute objectivity).

In the second, cultural sense objectivity is in effect redefined as social, as I argue in Ernest (1998) drawing on the social theory of objectivity proposed by Bloor (1984), Harding (1986), Fuller (1988) and others. This is how social constructivism views mathematical objects and truths. Such a perspective has a strong bearing on the discussion of the certainty of mathematical knowledge because it posits that mathematics is wholly located in the cultural domain and at least some of it is contingent on human history and culture.

The controversy between the traditional absolutist philosophies of mathematics and the maverick philosophies can be largely captured in terms of these two concepts of objectivity. One consequence is that both of these schools of thought can be said to acknowledge the certainty of mathematical knowledge although this has a different meaning according to the interpretation of ‘objectivity’. Mathematical knowledge consists of those mathematical propositions that are objectively warranted as true, or at least as logically valid, and hence can be claimed to be known with certainty.

Mathematical warrants are strong, reliable and promote a belief in the certainty of mathematical knowledge. Indeed mathematical warrants are among the strongest for any type of knowledge, since they are not subject to the errors or uncertainties arising from the use of empirical observation and testing against the phenomena of the physical world. But even if mathematical knowledge is infallibly certain, why do so many think that it is? Where do such strong beliefs come from? One of the innovations introduced by the strong programme in the sociology of knowledge is to treat true and false beliefs symmetrically (Bloor 1991). Thus it is not just false beliefs that need explanation. So too do true beliefs. Thus it seems both necessary and appropriate to explain where the belief in the certainty of mathematical knowledge comes from, what is its source.

**CERTAINTY IN MATHEMATICS AND ITS SOURCES**

To address the problems of the certainty of mathematical knowledge I pursue three lines of enquiry. First, to look at the historical development of mathematics to see how the cultural belief in its certainty has been constructed historically. Second, to briefly sketch individual cognitive development in mathematics to identify and highlight the sources of personal belief in the certainty of mathematics. Third, to examine the epistemological foundations of certainty for mathematics and investigate its meaning, strengths and deficiencies. Thus my first two lines of enquiry are intended to reveal how beliefs in the certainty of mathematical knowledge originate for society and individuals. Whereas my third line of argument aims to show the philosophical limitations of mathematical knowledge and the different ways that its certainty can and ought to be understood.

These enquiries draw on the history and sociology of mathematics, the psychology of learning mathematics and epistemology and the philosophy of mathematics. The interdisciplinary area of mathematics education is one of the very few if not the only area of knowledge where access to these different disciplines can be gained and a legitimate cross-disciplinary argument can be formulated.

**The historical construction of certainty in mathematics – its social origins**

1. **The invariance and conservation of number**

Mathematics as a discipline was formed about five thousand years ago with the development of systematic numeration systems and scribal schools teaching mathematical computation and problem solving in Mesopotamia and Egypt (Høyrup 1980). Mathematics had ritual functions such as in the construction of altars, but it is understood that its uses in recording and accounting for taxation and trade triggered its primary development. Through this development numeration systems were created and elaborated for accounting, record keeping, taxation and trade in support of kings, rulers and religious leaders. Thus numeration systems were required to be invariant with respect to the processes of counting and the products of numerical operations. Otherwise accounts, trade agreements, taxes, etc would not be recorded and enacted in stable and fair ways that could be trusted and relied upon by all parties, that is, would perform the required social functions. Indeed, according to Høyrup (1994), in ancient societies the reliability of calculation, measures and numerical records was also understood as part of the idea of justice, taking on ethical as well as utilitarian and ultimately epistemological value. Thus at the heart of systems of numeration and measurement is the human requirement that processes of accounting should conserve the material resources being recorded, and hence, by proxy be invariant with respect to the quantities, numbers and calculations involved.

The idea of conservation came into prominence as a very important concept in the nineteenth century in the physical sciences as employed in principles of the conservation of mass and energy. Conservation was also picked out by Piaget (1952) as a very important threshold concept in the child’s development of number sense, which I shall discuss further in the sequel. However, in mathematics it is taken for granted that conservation is a *sine qua non* of arithmetic. In the domain of number it is such a basic condition required in any meaningful application that it is not discussed. What is assumed as a fundamental basis that arithmetic rests upon is therefore not seen as a condition imposed by humanity to enable arithmetic to serve its original social function. In higher mathematics the concept of invariance is a prominent one but this concerns the structural properties conserved by mathematical operations and transformations in much more complex domains.

I wish to claim that the idea of counting material objects is not a ‘naturally’ given one, as simple and obvious as it looks to the trained modern eye. “It is really an ad hoc assumption to suppose that we have before us the universe of things divided into subjects and predicates, ready-made for theoretical treatment.” (Bernays 1935: 16). According to my analysis counting is based on a set of prior conceptualizations of the world that include the following five assumptions. These need to be taken for granted before counting and calculation can take place.

1. The world, or at least that part relevant for numeration, is understood to be made up of objects that are permanent or semi-permanent entities which can be individuated and distinguished.

In the first instance the objects to be counted were material entities, each a single connected whole such as a sheep, loaf, ingot, basket of grain, urn of oil, etc. These are not naturally given as distinct unitary objects existing in the world but are a product of the way that we first manipulate and then conceptualize and describe the world. Such a conceptualization results from years of socialization and training, from early childhood on. Once socialized, a person counting and calculating will most likely regard this division of the relevant parts of the world into discrete entities as a brute fact of nature as opposed to an imposed conceptualization (Ingold 2012).

1. Objects as we perceive them fall into kinds, and for the purposes of counting and accounting objects are understood to be interchangeable within kinds; treatable as equivalent units.

It is our conceptualization of the world that makes what we see and experience as events and entities individual and distinct and also makes some of them identifiable with each other and interchangeable. Thus, in the social world many individual objects are described as falling into kinds, such as sheep, loaves, etc. Objects of the same kind are understood, for the purposes of accounting, to be interchangeable and equivalent.

1. Objects of a single kind can be physically grouped into unified collections. Beyond this, we abstract our ideas to include collections that are not physically grouped. Consequently the idea of collecting objects together extends both to multiple kinds within a single collection and to purely abstract collections that are conceptually based and not physical grouped.

All collections, physical or conceptual are themselves conceptual objects that represent designated arrays of objects or events in the physical world, and are not the physical entities themselves. The abstract nature of collections allows great freedom in selecting which objects we wish to count and perform accounting operations on.

1. Processes of counting can be applied to any collection, but during the process the collection is viewed as a constant conceptual entity free from change, or else the process of counting is invalidated.

All of the physical world is in the process of flow and change, but for counting and accounting purposes the objects of our interest both individually and as a collection are timeless. This is achieved by dealing exclusively with conceptual objects and our conceptions remain static and timeless throughout all of our imaginary operations on them.

1. Any collection of objects can be counted resulting in a constant and invariant number, that is a fixed cardinal.

This property emerges from the specific conceptualizations and operations that we have constructed in the preceding assumptions and conceptualizations. Indeed it is with the goal of to achieving this property that these assumptions are made. We require the property of invariance for counting to serve its social purposes. This follows from the construction of counting, both culturally and individually, as an activity with certain specific properties (Gelman and Galistel 1978).

The above assumptions are also extended to continuous as opposed to discrete objects, such as an area of land, a heap of grain, a container full of oil or wine, etc. These are viewed by as permanent or semi-permanent entities which are decomposable into multiple units, that is, as collections of permanent or semi-permanent equivalent notional objects. These are then conceptualized and treated as discrete objects for counting.

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On the basis of these assumptions any collection of objects can be counted. Furthermore, such collections can be partitioned and recombined without any loss of the objects involved, and hence the numbers and quantities involved are invariant. Consequently, these seemingly innocuous assumptions provide a foundation for an arithmetic that necessarily conserves number under the usual operations of counting and calculating. This includes the addition and multiplication of numbers, and the usual properties of the operations are established, including the commutativity and associativity of the operation of addition (see Appendix A for details).

The counting and recording of collections of objects based on these conceptualizations enables records and receipts to be inscribed within the semiotic realm created by these numerical signs and operations. This semiotic realm produces an abstract world inhabited by unchanging meanings since collections and counts are understood as invariant. This world is timeless in another way too, because all operations are reversible without any variations in the salient properties and meanings of the signs. From these origins the semiotic realm of number operations takes on a fixed, reliable and apparently objective character. It is objective in the sense of existing independently of humankind, free from the processes of change and decay that characterize the lived world. These properties provide the foundations of certainty in mathematics. This certainty is based on the reliability and invariance of counting outcomes and calculations, which always provide the same objective answers provided the rules of procedure are correctly followed. Thus the mathematical accounting practices supporting taxes and trade are founded on reliable and trustworthy ideas and methods, transparent and fair to all. A consequence is the founding of the shared belief in the certainty of mathematical methods.

During the first half of the history of mathematics, from circa 3000 BCE to around 500 BCE, calculation based on numbers and computation (including their applications in weights and other measures) was the dominant part of mathematics. While there was certainty about the outcomes of calculations, no written claims of certainty were made and correctness of rule following rather than the truth of mathematical statements was the focus of mathematical activity. Nevertheless, the culturally installed conceptualizations underpinning the reliability, invariance and objectivity of number and calculation provide the original foundations of certainty for mathematical knowledge and for a belief in the certainty of mathematics.

**2. The ontology of number**

During the course of the first half of the history of mathematics the conceptualization of number developed and changed. Although it is hard to know how the Sumerian and ancient Egyptians conceptualized and understood number, it is very likely that the conceptions changed as symbolic numeration and calculation practices emerged, developed and became routine. For calculations would be conducted by symbolic procedures and manipulations within the symbolic system rather than by reference to the entities being counted or operated upon mathematically. Consequently number meanings would be extended to include numerical representations, i.e., numeral notations. It is likely that ancient Greek conceptions of number changed significantly between the 6th and 3rd centuries BCE. Wilder (1974) claims that it took a long time for number to evolve from being understood as adjectival (e.g., three sheep) to numbers as nouns (e.g., three). He suggests that number mysticism played a significant part in this evolution, with numbers such as seven having special powers (luck, sacredness) attributed to them, independently of what they were used to quantify. Aristotle claims that even as late as the time of the Pythagoreans numbers were not separable from the things they were used to quantify (Klein 1968)[[2]](#footnote-2). However, by the time of Plato three centuries later numbers were conceptualized as self subsistent entities existing in an ontological category apart from things bodily and mundane. Plato not only nominalised many adjectival words (e.g., three, true, beautiful, good) but created an ontology of ideal abstract forms in a separate platonic reality to represent their meanings (Three, Truth, Beauty, The Good). As timeless self-subsistent forms existing unchanged and unchanging in a Platonic realm independently of humans these ideas, especially numbers, became entities that could be known with certainty.

Historically, platonism has not been the only ontology of number and universals. For example, the scholastic philosophers of the medieval period contrasted realism with nominalism and conceptualism. However, platonism retains a dominant position in the philosophy of mathematics, with nominalism and conceptualism surviving under the banners of formalism and intuitionism, respectively, as well as in other views.[[3]](#footnote-3) The widespread adherence to platonism by modern mathematicians and philosophers underpins belief in the objective and independent existence of mathematical objects, and hence provides a basis for a belief in the certainty of mathematical knowledge (Cohen 1971). For if mathematical objects have an objective and independent existence so too do their properties and the relations that hold between these objects in their abstract world. Consequently, since mathematical knowledge is seen to comprises facts describing these objective relationships, observable or at least verifiable in the realm of mathematical objects, it provides a solid and independent basis for a belief in its certainty.

**3. The role of proof**

In accounts of the second half of the development of mathematics, from circa 500 BCE to the present, historians and philosophers of mathematics foreground the importance of proof in the discipline of mathematics (Boyer 1989). Indeed, the Ancient Greek introduction of proof into the Western intellectual tradition is sometimes heralded as the dawn of ‘real mathematics’.

Proof is now so commonly taken for granted as the very spirit of mathematics that we find it difficult to imagine the primitive thing which must have preceded mathematical reasoning. (Bell 1953: 21)

Here by reasoning Bell is referring to deduction as opposed to the reasoning involved in problem solving and the application of methods and algorithms that underpinned and long preceded the rise of deductive proof in mathematics. However, the differences between these two forms of reasoning have been exaggerated in accounts of the history of mathematics (Ernest 2007). Some of the overvaluing of proof over computation is due to a failure to see the strong analogy between the two forms. Some of it is due to an Eurocentric ideology that has dominated historical and philosophical thought for the past two hundred years. This ideology elevates rationality based on reason in the narrow deductive sense as the highest intellectual good. Bernal (1987) has argued that during this period ancient Greece with its introduction of axiomatic proof has been ‘talked up’ as the starting point of modern European thought, and the ‘Afro-asiatic roots of classical civilisation’ have been neglected, discarded and denied. Thus the vital developments in number and calculation in Mesopotamia, ancient Egypt and Asian civilizations are unrecognized as the essential foundation for all of mathematics including proof, as well as a central pillar of mathematical reasoning from ancient times to the present.

Mathematical proof and calculation are formally very close in structure and character. Mathematical topics from informal areas (e.g., number and calculation) to axiomatic theories (e.g., Euclidean geometry) can all be represented as semiotic systems (Ernest 2008). In any of these areas derivations can be represented as finite sequences of signs in which each sign is derived from its predecessor in a rule-following way. Such sequences can represent a deductive proof for a theorem as a sequence of sentences, each derived from its predecessors by the deductive rules of the system. The final sentence is the theorem proved. The rules of proof employed in the sequence are based on the preservation of the truth value of sentences in each deductive step, and hence along the length of the proof sequence to the theorem proved. Of course in many published proofs some of the proof steps are based on implicit rules, known through their instances and applications. But in each case they are understood to be truth-preserving.

In the case of a calculation, the initial sign of a calculational sequence is usually a compound term. Subsequent terms are derived by the rules of calculation and typically each is a simplification in some sense of its predecessor. The final term in a calculation is the simplified numerical solution to the problem. Thus calculations are sequences of terms, each derived from predecessors by the rules of the system. The rules of calculation are based on the principle of the preservation of numerical value.

Thus there is a strong analogy between calculation and proof. The rules applied in each are based on the principle of value preservation, whether it be numerical or truth value. In addition, terms and sentences are structurally very similar, each defined analogously by induction. Both begin with elementary or atomic signs, from which compound signs are constructed by means of operations/functions or logical connectives, respectively, in stages of increasing complexity. Furthermore, proof and calculation are formally equivalent, in modern foundational terms. Every calculation can be represented as a deduction of identities, and every proof can be represented as a sequence of terms (Ernest 2007).

The sequential and rule-based nature of calculation is something that precedes the development of the deductive proof of theorems by well over a thousand years. My contention is that without the long and ancient tradition of rule following in sequences of calculations, without confidence in the reliability of its steps, and without the entrenchment of its representational and value preserving features, the development of proof would not be possible. The striking analogy between calculations and deductive proofs puts into question the claimed superiority of proof. However, my purpose here is not primarily to mount an ideological critique of the Eurocentric history of mathematics and its philosophical parallels, but to point to calculation and proof as two constituents in the development of belief in the certainty of mathematics.

Beyond this analogy, it is also important to note the difference between proof and calculation. Calculations are procedures conserving numerical value and giving reliable results that provide a foundation for the subsequent development of certainty in mathematics. However, proof also incorporates a strong dialogical element (Ernest 1994). The persuasion of others plays a large part in the development and purposes of proof. For persuasion is the attempt at communicating the truth of a claim, that is to convince others of its certainty.

Proof is central in modern mathematics in persuading mathematicians and others of the certainty of mathematical knowledge. For the past two and a half millennia mathematical proofs have been persuasive demonstrations of mathematical claims. Some, like the irrationality of root 2, or the infinite number of primes, have been short, pithy and persuasive arguments for the certainty of their end point. A series of short steps, linking one claim to another, leads to a surprising overall conclusion, one that was not at all evident at the outset. Clearly such proofs, and proofs in general, are very powerful means of convincing their readers of the certainty of their claims. Proofs in general constitute the strongest evidence for the certainty of mathematical knowledge. However, that the perception of proof as a warrant for mathematical truth and certainty is not a natural, given reaction. The apprehension of proof requires many years of training and mental cultivation along specific cultural paths. I will elaborate on this below. My point here is that proofs as means of persuading their readers or listeners of the truth and validity of their conclusions, constitute a further vital step in the construction of cultural beliefs in the certainty of mathematical knowledge.

**4. Historically mathematics engulfs uncertainty**

The history of mathematics is not only a trajectory in which the methods of mathematics are refined and developed with increasing precision to conserve value, reliability and truth, laying the groundwork for belief in the certainty of mathematical knowledge. In addition, sources of uncertainty arising within mathematics or in areas of thought and application outside of mathematics are colonised and appropriated within mathematics. This tames and routinises them so that they are accommodated within the overall narrative of mathematical control, predictability and certainty. Indeed, the history of mathematics can read as a narrative of such engulfment. Mathematics has been defusing uncertainty by colonising it since its beginnings, starting with the incommensurability of lengths as shown by the irrationality of root 2 among the Pythagoreans, if not earlier.

The term irrationality is illuminative here. Ratios of the form p:q between whole numbers, including, equivalently, fractions defined as p/q, demonstrate the commensurability of lengths, since both p and q are based on a shared unit, namely unity or 1. Such relations of commensurability are termed rational, and by metaphorical extension gives rise to the modern broader conception of rationality; that is, analyses and arguments based on shared concepts and logical reasoning. Just as rational numbers must be expressible as a ratio of whole numbers p:q based on a shared unit, so too rational reasoning depends on a shared basis of logic and principles of reasoning among interlocutors, upon which opposing arguments can be shown to be based.

Historically, the discovery that the simple diagonal of a unit square is irrational reportedly caused great distress among the pythagoreans (Kirk and Raven 1957). The square root of 2 is incommensurable (with the length of the side of the unit square) because it cannot be represented as the ratio of any two whole numbers. The pythagorean belief in a simple narrative of rational order and certainty was shown to be false by this discovery. Their view, that whole numbers and their simple relations sufficed to describe both the empirical world and the semiotic world of mathematics, was torpedoed. Interestingly, it was one of the first recorded proofs that challenged this view. For the assumption that root 2 can be expressed as a rational leads, in a few deductive steps, to a contradiction. However, mathematics did not crumble. All that happened with that the concept of length (and later that of number) was broadened to incorporate incommensurable lengths, i.e., irrationality in its original sense. That was over two and a half millennia ago, at the birth of mathematical proof.

The crisis over irrationals was not an isolated incident. The history of mathematics is peppered with problems arising from the introduction of new concepts, theories, methods and results, each challenging the boundaries of mathematical acceptability. These include Zeno's paradoxes of motion, the insolubility of the delian problems, the introduction of negative numbers, the theory of probability (the first science of unpredictability), the calculus with its infinitesimal numbers, imaginary and complex numbers, non-Euclidean geometries, Hamilton’s rejection of commutativity in algebra, the doubly incommensurable transcendental numbers, statistics (the second science of unpredictability), Cantor's set theory with its different sized infinities, counterintuitive functions including Peano’s and others’ space filling curves, logical paradoxes, Gödel’s incompleteness theorems, the independence of the continuum hypothesis, uncheckable computer proofs such as the 4 colour theorem, catastrophe theory, and fractals and chaos theory. The introduction of each one of these elements caused philosophical anxiety and controversy. Each new topic challenged the predictability and certainty of mathematics. However, mathematics appropriated, routinised and instrumentalised each of these enlargements so they were integrated into new theories, simply adding to the ‘toolbox’ of mathematics, and thus posing no threat to its certainty. Despite the temporary anxiety during their introduction, all of these topics were and remain widely accepted as technical and conceptual advances, and ultimately not as challenges to the underlying paradigm of rational control and scientific certainty. Chaos is only the latest branch of mathematics to be tamed and engulfed, and not the beginning of a wholly new game. Gödel’s Theorem, now over 80 years old, can be seen as far more significant, as it reveals structural flaws in the foundations of mathematics on which much of its claimed certainty rests. Nevertheless, mathematics did not even break stride with its discovery as it stepped over and encompassed this and other limitative results. Mathematics just continued advancing, generating and incorporating new knowledge.

There is an interesting parallel with modern art. Impressionism, expressionism, cubism, suprematism, futurism, abstraction, constructivism, dada, surrealism, abstract expressionism, pop art, minimalism, conceptual art, Brit art, and postmodernism are all art movements that have challenged the boundaries of art, artistic representation and good taste. They have caused outrage, furore, and cries of condemnation, rejection and denial that they are art. But art continues unabated and the museums and dealers continue to show, buy and sell the products of these movements without qualms, and with increasing monetary values. The practices of art, the making, showing, selling, collecting, and reviewing of art, like the practices of mathematics, continue without slackening their pace.

Lyotard (1984) considers all of human knowledge to consist of narratives, whether in the traditional narrative forms, such as literature, or in the scientific disciplines. Each disciplined narrative has its own legitimation criteria, which are internal, and which develop to overcome or engulf contradictions. Lyotard describes how the discipline of mathematics overcame the crises in the foundations of axiomatics brought about by Gödel’s Theorem in this way, by incorporating meta-mathematics into its enlarged research paradigm. He also noted that continuous differentiable functions were losing their pre-eminence as paradigms of knowledge and prediction, as mathematics incorporates undecidability, incompleteness, catastrophe theory and chaos. Thus, he concludes, a static system of logic and rationality does not underpin mathematics, or any discipline. Rather disciplines rest on narratives and language games, which shift with the organic changes of culture. Lyotard claims that the traditional objective criteria of knowledge and truth within the disciplines are internal myths that attempt to deny the social basis of all knowing. This postmodern perspective, like a number of other intellectual traditions, affirms that all human knowledge is interconnected through a shared cultural substratum, and that it is a social construction.

Thus despite the upheavals, changes, expansions and continued growth of mathematical concepts, methods and knowledge, belief in the reliability and certainty of mathematics continues unabated. The theoretical form of mathematical knowledge as a deductive edifice based on certain axioms and methods, first epitomized in the elements of Euclid, continues to be used in the attempt to give certainty to disparate knowledge systems. Newton’s Principia, Descartes’ epistemology, Spinoza’s ethics, Napoleonic law, the American declaration of independence, and even Alistair Crowley’s (1929) treatise on black magic utilize the axiomatic form in order to convince their readers of the truth and certainty of their systems.

Overall, my point is that when confronted by historical discontinuities, breaches of accepted rules, paradoxes, antimonies and even contradictions, mathematics with its deductive form remains the paradigm of certainty, with initially disruptive innovations and developments rapidly incorporated into the discipline as further tools for the production of certainty.

**The individual and certainty in mathematics**

The development of personal knowledge in the individual in some ways parallels the historical development of knowledge. This was noted by Lubbock (1865: 570) “The life of each individual is an epitome of the history of the race, and the gradual development of the child illustrates that of the species.” In the realm of mathematics this parallel was elaborated by a number of authors such as Branford in 1908 (Fauvel 1991). In psychology the insight was further developed by Vygotsky as a central feature of his psychological theory.

Every function in the child's cultural development appears twice, on two levels. First, on the social and later on the psychological level; first between people as an interpsychological category, and then inside the child as an intrapsychological category. (Vygotsky, 1978: 128)

In learning mathematics, whether in school or out of it, children meet numerals and numerical procedures through a range of social practices that affirm the constancy, reliability, and invariance of number and quantity. Through this experience, a belief in the reliability of counting and calculating, and hence of the certainty of mathematics is established. According to Vygotskyan theory, this view of mathematics first encountered socially becomes internalised and forms part of the child’s outlook and functioning.

Further details of this are accounted for in the conceptual development of the child’s the understanding of mathematics. As was noted above, in Piaget’s (1952) theory a child’s development of mathematical concepts passes through the crucial stage of conservation, when it is understood that number is conserved irrespective of the order, appearance and rearrangement of the items being counted. This stage also encompasses at different times the understanding of the conservation of other measures and quantities such as area and volume. According to Piaget, after achieving this understanding of conservation children move on to the stage of concrete operations in which it is understood that mathematical operations are reversible. Piaget describes this in terms of mental operations corresponding to mathematical ones which can be played backwards and forwards in the imagination at will. Operations such as addition and set-union can be applied or reversed while still conserving overall numerosity. These capacities correspond to reversibility properties in the historical development of counting systems, as described above.

According to Piaget conceptual development progresses to the next stage, termed formal operations, typically when children are at secondary school, from age 11 years onwards.[[4]](#footnote-4) At this stage children are able perform formal symbolic written work with understanding. Thus according to Piaget work on ratio and proportion, algebraic reasoning, and logical deduction require the child to have reached the stage of formal operations. During this stage it is theorized that are not only are children able to perform and understand formal symbolic operations, but that their understanding changes with invariance and reversibility understood to apply to more abstract mathematical ideas.

Different post-piagetian theories of students’ conceptual development exist, but a common theme is that the understanding of concepts as processes is transformed into a understanding of them as objects. Sfard (1991) argues that this is a shift from process from operational to structural conceptions, and that three steps occur: *interiorization*, a process with familiar objects, *condensation*, where the former processes become separate entities and *reification* in which students *“*see this new entity as an integrated, object-like whole.” (Sfard, 1991: 18). Another theorist Dubinsky (1994) proposes an extended sequence in his APOS theory, according to which learners construct mental *actions*, *processes*, and *objects* and organize them in *schemas* in their development and use of mathematical concepts. Both of these theoretical approaches build on Piaget’s theory of cognitive development by adding a stage in which the understanding of mathematical concepts shifts from seeing them as processes or actions to seeing them as self subsistent objects in their own right, to which higher level processes can be applied. Once again this parallels developments in the history and development of mathematics. Furthermore, Sfard’s account is compatible with Vygotsky’s ideas of intellectual development.

The outcome of these individual processes of concept development is the construction of a personal ontology of mathematical objects in which these objects seem to have an independent existence on their own. To this belief can be added the certitude of outcomes of mathematical procedures experienced and performed over many years. Most mathematical tasks undertaken by learners have unique right answers. Even when more sophisticated open-ended tasks are performed, the constituent procedures have unique correct outcomes. It is the learner’s choice of strategies and procedures to apply that make the tasks open, not any arbitrariness in acceptable answers. These experiences contribute to a belief in the certainty of mathematics. To such sources of conviction there is the further certainty vouchsafed by mathematical proof as it is understood in the latter part of schooling. The result is a well-buttressed belief in the certainty of mathematics. As this account shows, such belief is not arrived at overnight, but is the end point of a process of engagement with mathematics lasting upwards of ten years in school alone. For advanced students of the field, this is followed by half a dozen years of intense engagement with mathematics in college and university. During this engagement belief in the certainties of the outcomes of mathematical calculations and proof procedures is further strengthened.

In these processes, the perception of proof as a warrant for mathematical truth and certainty is not something natural, independent of culture. A belief in the certainty of mathematical knowledge is not one that emerges ‘naturally’ in a developing person, but is something that derives from many years of engagement with the subject and associated cultural presuppositions. Belief in the certainty of mathematics is constructed by the individual as a response to an extended and highly directed and shaped experience of learning and doing mathematics. In this respect it resembles the historical development of mathematics. Both are long sequences of development in which concepts evolve and become more abstracted and solid in an idealised way. Individual development also overcomes and engulfs uncertainty as the discipline has done. Piaget applies the term accommodation to the process whereby conceptual frameworks are restructured and enlarged to overcome limitations and contradictions in individual understanding as they encounter problems of growing complexity.[[5]](#footnote-5) This parallels what are termed revolutions in the history of mathematics and science, as these fields renew themselves to overcome theoretical limitations and contradictions (Ernest 2013, Kuhn 1970, Gillies 1992).

Accounting for the development of belief in the certainty of mathematical knowledge both historically and individually does not bring into question its validity. A true belief can be derived in this way just as well a misleading one might be. What this account reveals is the mechanisms whereby such beliefs are constructed culturally, rather than being forced on us simply by the fact of their truth.

Belief in the certainty of mathematics is almost inescapable, based on the causal factors in the two domains that I have sketched. First there are the cultural and disciplinary developments concerning mathematical knowledge, with four contributing factors.

* 1. The conceptualization and assumptions underpinning arithmetic and mathematics that require the invariance and conservation of number and calculation, and that go on to produce the reliability of calculation almost as a by-product.
  2. During the development of mathematics the ontology of number solidifies and crystallises out. There is a move from numbers as quantitative adjectives to numbers as seemingly independent self-subsistent entities existing in an unchanging abstract realm with invariant properties and independent and objective existence.
  3. A significant factor in the history of mathematics is the emergence of proof with its goal of warranting the truth of mathematical propositions with certainty. This plays a major role in convincing readers or listeners of the certainty of mathematical propositions.
  4. Within mathematics, as in any domain of knowledge, developments introduce contradictions and uncertainties into the discipline. Almost uniquely mathematics is able to colonise, neutralise and appropriate these developments and incorporate them, without having to accommodate its overall paradigm, into its narrative of certainty.

These factors underpin the shared cultural and historical beliefs in the certainty of mathematical knowledge.

Second, the development of individuals, guided by the processes of schooling, provides enculturation into the conceptions, practices and beliefs of mathematics. These are internalized by students and almost invariably lead to a growing conviction in the certainty of mathematics over the long period of engagement.

Thus independently of the status of the certainty of mathematical knowledge, whether such claims are valid or not, I have shown how beliefs in this certainty emerge culturally, historically and personally. I now turn to the second question addressed in this paper: can mathematical knowledge be known with certainty? Can mathematical knowledge be claimed to be certain at the highest level and beyond all doubts?

**The problem of truth**

The question of the certainty of mathematical knowledge cannot be answered without addressing the problem of mathematical truth. What is the truth status of mathematical knowledge and the theorems of mathematics? Is mathematical knowledge true with certainty?

Language is a very seductive thing. As I have suggested above, since the time of Plato having words to describe abstract ideas seems to bring the entities named by such words into existence. Such entities create and populate an ideal world, a platonic realm, if they cannot be found in our everyday material world. I wish to claim that such a view represents an ideology gripping much of traditional philosophy. It sees language, and claims expressed as sentences as ‘mirroring nature’. Rorty (1979) critiques the traditional assumption that there is a given, fixed, objective reality and that text as well mind and knowledge capture and describe, with greater or lesser exactitude. This traditional ‘mirroring’ philosophy reached its apogee in Wittgenstein’s (1922) *Tractatus* with the picture theory of meaning. Wittgenstein’s early doctrine asserts that every true sentence depicts, in some literal sense, the material arrangements of reality. Language, when used correctly, floats above material reality as a parallel universe and provides an accurate map or picture of it. However, a claim this strong is hard to sustain, and Wittgenstein and even the logical positivists withdrew from this overly literal position about the relationship between language and reality. They adopted instead the verification principle which states that the meaning of a sentence is the means of its verification (Ayer 1946). For without this revised view of meaning the predictive power and generality of scientific theories is compromised. In his later philosophy Wittgenstein (1953) also wholly rejected this position himself having pushed the picture or mirror view to its limits in the *Tractatus*.

However, the mirroring ideology applied to mathematics remains very potent in Western culture. Mathematics is seen to describe an objective and timeless superhuman realm of pure ideas, the necessity of which is reflected in the ineluctable patterns and structures observed in our physical environment. The doctrine that mathematics describes a timeless and unchanging realm of pure ideas goes back to Plato, and many of the greatest philosophers and mathematicians have subscribed to the doctrine of Platonism in the subsequent millennia since the time of Plato. In the modern era this view has been endorsed by many thinkers including Frege (1884, 1892), Gödel (1964), and in some writings by Russell (1912) and Quine (1953). According to Platonism, a correct mathematical text describes the state of affairs that holds in the platonic realm of ideal mathematical objects. Mathematical sentences are nothing but descriptions or mirrors of what holds in this inaccessible realm. In other words, a mathematical truth is a sentence that truly describes what holds in this platonic realm.

This view has its seductions. Mathematicians and philosophers have a strong conviction of the absolute certainty of mathematical truth and in the objective existence of mathematical objects, and a belief in platonism validates this. However, platonism posits a mysterious realm without indicating how access to its objects and truths can be gained. Such access can only be gained directly or indirectly.

Direct access to the platonic realm must be by intuition. That is, the objects of mathematics and the truths of their relationships that make up mathematical knowledge are intuited. In intuition the ‘mind’s eye’ sees or otherwise perceives the existence of objects and the truth of mathematical knowledge statements directly. This may be enough to engender belief in the certainty of mathematical knowledge in the person experiencing it, but it is not an adequate basis to persuade others that all doubts must be rejected and that mathematical knowledge is to be accepted without any further warranting. The only way such direct intuitions could be persuasive to all would be if all had identical intuitions. But this is manifestly false, for not all share the same intuitions. This is well illustrated by the philosophy of mathematics called intuitionism. This promotes the view that the basis of mathematics is given by pure intuition (Brouwer 1913). But the majority of mathematicians and philosophers reject some of the so-called truths of mathematics put forward by intuitionists, showing that there is no consensus on the knowledge provided by mathematical intuition. Further, even if all mathematicians at any one time did agree on a shared mathematical intuition, this would not guarantee that such agreement would last forever. A shared belief needs an ironclad warrant to turn it into knowledge.

Since there is no direct access to the truths of mathematics, such access must be indirect. But indirect access to the mathematical truths of the Platonic realm of must be via reason or proof. If reason in the form of proof is to be used, then in order to establish the truth with certainty of mathematical knowledge the following conditions are a minimum requirement. We must have:

1. A starting set of true axioms or postulates as the foundation for reasoning;
2. An agreed set of truth preserving procedures and rules of proof with which to derive truths;
3. A guarantee that the procedures and rules of proof are adequate to establish all the truths of mathematics (completeness); and
4. A guarantee that the procedures and rules of proof are safe in warranting only truths of mathematics (consistency).

However, each one of these conditions raises problems.

1. It is not possible to warrant a starting set of axioms or postulates true indirectly, as this leads to an infinite regress (Lakatos 1962). Some assumed truths are required as a starting point in any proof, and as I have argued above, intuition is not enough to guarantee their truth. So the axioms and postulates must be assumed and mathematical proofs take on a hypothetico-deductive form. That is, theorem T is true assuming assumptions A are true. A entails T. This is acceptable but it means that mathematical truths are not absolute but relative to the set of assumptions made. By saying that ‘A entails T’ is our truth rather than ‘T’, we partly circumvent this.

2. Current mathematical practices exhibit a variety of accepted reasoning and proof styles. Exemplars of published proofs are accepted as valid by communities of mathematicians, based on professional expertise rather than explicit rule following. However, different mathematical specialisms require different proof styles and levels of rigour (Knuth 1985). As this suggests, none of the proofs published are fully explicit or fully rigorous, even with respect to the standards of the relevant sub-discipline (Kline 1980). My contention is that they cannot be made so. In Ernest (1998) I argue that most published proofs could not be translated into fully rigorous formal proofs with any guaranteed degree of safety. Even if they were, because of their sheer size rigorous formal proofs could not be checked for correctness with any degree of certainty either (MacKenzie 1993). Thus the rules of proof provide practical certainty, relative to the prevailing accepted proof practices rather than any absolute degree of truth and certainty.

3. It is accepted following Gödel’s (1931) first incompleteness theorem that in any but the simplest mathematical theories the rules of proof are inadequate for establishing all of the relevant mathematical truths. Thus most mathematical theories are incomplete in that there are truths of the system that are demonstrably unprovable. So mathematical provability does not capture mathematical truth and indeed falls some way short of it (Paris and Harrington 1977).

4. It is also well known following Gödel’s (1931) second incompleteness theorem that no guarantee can be given that rules and procedures of proof are safe in warranting only truths of mathematics. It is not possible to prove the consistency of any sufficiently complex formal theory using only its assumptions and axioms. Further assumptions are required for such a proof, assumptions that exceed those of the formal theory to be safeguarded. This does not mean that one cannot prove the consistency of any particular axiomatic system. Gentzen (1936) proved the consistency of Peano arithmetic using transfinite induction. This might satisfy a working mathematician, but philosophically it remains problematic, for using ‘intuitively obvious’ non-finitistic principles in addition to some of the axioms of the system and logic means that more has to be assumed than is safeguarded.

What this shows is that the truth of mathematical knowledge cannot be shown with absolute certainty. It cannot be said, free from all caveats and restrictions, that mathematical knowledge is absolutely true. Proofs in general constitute the strongest evidence for the certainty of mathematical knowledge, and mathematical proof lies at the heart of claims of mathematical certainty. It is the strongest weapon in the armoury used to persuade others of the certainty of mathematical knowledge. But proof cannot guarantee the absolute truth of mathematical knowledge.

Is it surprising that I reject hubristic claims of certainty for mathematical knowledge extended across all time and space, across all possible and undreamed of universes? Should we be entitled to claim that because human reason and rationality cannot imagine any circumstances in which other conscious beings might refuse to be convinced by our claims, assuming we could make them understood in the first place, that these claims have universal certainty? Dare we assume that the reasoning and knowledge achieved by a single species of animals living in a small corner of an unimaginably large universe is universally valid, necessary with certainty throughout eternity and to infinity and beyond? In my opinion, such a claim is unfounded and arrogant, and in effect attempts to elevate our knowing to the unimpeachable level of the gods. Both believers and non-believers regard such claims as near blasphemous (Howell and Bradley 2001)

The conclusion that mathematical knowledge cannot be claimed to have absolute certainty is shared by many of the leading mathematicians, logicians and philosophers of the past hundred years. In Ernest (1991, 1998) I quote similar insights from Hersh, Kline, Lakatos, Polanyi, Popper, Putnam and Wittgenstein. In his paper 'A renaissance of empiricism in the philosophy of mathematics' Lakatos (1978) quotes to demonstrate their common view concerning the impossibility of complete certainty in mathematics from the later works of Russell, Fraenkel, Carnap, Weyl, von Neumann, Bernays, Church, Gödel, Quine, Rosser, Curry, Mostowski and Kalmar; a list that includes many of the key logicians of the twentieth century.

In addition to Gödel’s incompleteness theorems mathematics has other limitative results including Tarski’s proof of the undefinability of truth, the Lowenheim-Skolem theorem about the uncontrollable size of models, Church’s theorem on the undecidability of validity and Craig's Interpolation Theorem on the impossibility of expressing proofs in any final ultimate form. Rather than seeing these as signs of weakness such theories and results should be celebrated as advances in knowledge that come from an understanding of the limitations of human knowing. To know the limits of your knowledge is wisdom, not ignorance.

There is an illuminating analogy with developments in modern physics. General relativity theory requires relinquishing absolute, universal frames of reference in favour of a relativistic perspective. In quantum theory, Heisenberg's uncertainty principle means that the notions of precisely determined measurements of both position and momentum for particles has also had to be given up. The position of all particulate matter is given by probabilistic functions and cannot be precisely located as it could be in obsolete mechanical models of the universe. But what this means is not the loss of knowledge of absolute frames and certainty. Rather it represents the growth of knowledge, bringing with it a realization of the limits of what can be known. Relativity and uncertainty in physics represent major advances in knowledge, advances which take us beyond what was known, leading to new understandings, predictions and applications. They take us to the limits of knowledge, at least until they are replaced by new theories, as inevitably happens in science.

Likewise in mathematics, as our knowledge has become better founded and we learn more about its basis, we have come to realize that the absolutist view is an idealization, a myth. But this too represents an advance in knowledge, not a retreat from what was viewed as the stronger position of absolute certainty of the past. Mathematics remains the best grounded and most certain area of knowledge or of anything made by humankind. Relinquishing the hope of absolute certainties (Kline, 1980) does not represent a loss of knowledge. It was Vico’s (1710) great insight that we can only know with certainty that which we have made.

Verum is a priori truth, and is attained in, for example, mathematical reasoning, where every step is rigorously demonstrated. Such a priori knowledge can extend only to what the knower him[/her]self has created. It is true of mathematical knowledge precisely because [humans] themselves have made mathematics. It is not, as Descartes supposed, discovery of an objective structure, the eternal and most general characteristics of the real world but rather invention: invention of a symbolic system which [humankind] can logically guarantee only because [humans] have made it themselves, irrefutable only because it is a figment of [humanity]'s own creative intellect. (Berlin 1969: 371)

Mathematics is one of the great human inventions, constructed and reconstructed over millennia. Our certainty in mathematics resides in knowing what we have constructed. What this means is that we have to be circumspect in what we claim in terms of mathematical certainty and objectivity. Mathematical knowledge is known with certainty within the bounds of what can be humanly known. Likewise mathematical knowledge and mathematical objects are objective, but in the circumscribed sense of cultural objectivity defined above. We do not have to relinquish the certainty and objectivity of mathematics. We merely have to be more circumspect in what we claim by ascribing certainty and objectivity to mathematics.

**Conclusion: the problems of mathematical certainty**

In this paper I have distinguished two problems of certainty. First, there is the question of whether mathematical knowledge is known with certainty. I have argued that the absolute notions of truth and certainty are beyond humanity’s reach. We can know mathematics with certainty, but certainty and objectivity in all of our knowing is circumscribed by the limits of human knowing. We are not gods and cannot claim to touch the absolute in our knowing. As the social world and human culture change, so too there is the possibility of change to some of what we believe we know in mathematics with certainty. One way this has happened in the past is that our ideas and standards of truth, provability and hence of certainty have changed over the history and development of mathematics. It cannot be said that mathematical knowledge is absolutely true. Proofs in general constitute the strongest evidence for the certainty of mathematical knowledge. Mathematical proof lies at the heart of claims of mathematical certainty. It is the strongest weapon in the armoury used to persuade others of the certainty of mathematical knowledge. But proof cannot guarantee the absolute truth of mathematical knowledge absolutely and eternally. Within the circumscribed domain of human knowing, of culturally objective knowledge, mathematics is known with certainty.

Secondly, there is the problem of explaining why it is so widely believed that mathematical knowledge is known with certainty. Such a belief has to be constructed, and the fact that we accept that it is true does not explain the origin of such a belief. In accounting for beliefs in the certainty of mathematical knowledge I have provided arguments from two directions: the social and cultural domain, and the territory of individual psychological development. From the former direction I have shown how the historical development of mathematics produces a belief in the objectivity and certainty of mathematical knowledge. This belief emerges because the origins of mathematics in counting and calculation require invariance, predictability and reliability in order to fulfil their social purposes. In addition, familiarity with numbers and other mathematical objects over the millennia has led to a belief in them as independently existing stable and invariant objects with fixed and enduring properties that can be known with certainty. I have also argued that mathematics engulfs, tames and appropriates any troubling concepts and uncertainties. Even seemingly contradictory ideas and methods that have emerged as challenges to the smooth running of mathematics have been turned into routine and predictable new theories. Thus belief in the certainty of mathematics is not threatened by new developments. The emergence of proof as a warrant for mathematical knowledge is the final historical dimension ensuring a belief in the certainty of mathematical knowledge in society. For proof is nothing more than a means of persuasion that mathematical results are true and known with certainty.

My second direction of approach has been to examine elements of an individual’s psychological development. This, which in some respects parallels this historical and cultural development of mathematics, shows how learning mathematics and many years of working with mathematical signs, processes and problems produces a belief the objectivity of mathematical objects and in the certainty of mathematical knowledge.

These two strands, the historical and individual, offer an account of how the belief in mathematical certainty has been constructed, partly independently of its validity. However, some of the most potent aspects of the historical development of mathematics, such as the invariance and reliability of number and calculation, and the emergence of the persuasive power of proof, also serve to inculcate personal beliefs in the certainty of mathematics among individuals.

Thus I claim that the two problems of certainty identified and distinguished above have been provisionally solved. Mathematics is a part of knowledge that is known with as much certainty as any other human knowledge. Indeed, mathematics remains the best warranted domain of all human knowing. Beliefs in the certainty of mathematics are widespread and deep. By distinguishing the absolute certainty which I challenge and refute, from that which is circumscribed by the limits of human knowing, I am able to distinguish a form of mathematical certainty consistent with social constructivism and other maverick philosophies of mathematics. This respects the epistemological depth and strength of mathematical knowledge warrants. It also honours mathematics as one of the greatest and most awe-inspiring achievements of human intellect and ingenuity. Mathematics can be known with certainty and beliefs in its certainty are justified and warranted.[[6]](#footnote-6)

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**Appendix A:**

**Operations on collections and the resultant properties of arithmetic**

The basis for counting and arithmetical operations, as I have defined them, depends on the conceptualization of unitary entities (objects) and collections of objects. Collections can be said to correspond to sets, as developed in modern mathematics. Consequently, the objects in any finite collection can be counted and the result is an invariant number. As we now understand it, counting is the assignation of the sequence of ordinal numbers to a finite set of objects (in 1-1 correspondence) which results in a final unique ordinal. This provides a cardinal number, which is a unique and invariant property of the collection and of any other equipotent set.

Historically, oral counting developed prior to the written arithmetic being discussed here. Thus adjectival oral numbers were available for ordinal and cardinal purposes when written numeration and calculation were first developed around five millennia ago.

In offering a rational reconstruction of early arithmetic using modern set theory, we can use the modern definition of the finite ordinal numbers for clarity. The standard definition by induction is as follows. The first ordinal is 0 =def ∅. Given ordinal n, we define its successor n’ =def {{n}∪n}. Thus 1 =def {{0}∪0}={{∅}∪∅}={{∅}). 2 =def {{1}∪1}={{{∅}}∪{∅})={{{∅}}, {∅}}, etc. Note that any number n is given by some set N, such that C(N)=n.

The cardinality of a set is the number of distinct elements in it. Technically, the cardinality C(S) of a set S can be defined inductively. C(∅)=0. If C(S)=s, and x∉S, then C(S∪{x})=s’, where s’ is the successor of n in the sequence of ordinals

The operation of numerical addition of two (cardinal) numbers is based on taking the cardinality of the combined set, the union of two distinct collections with cardinalities equalling the two corresponding numbers. Thus, if s and t are cardinal numbers, and S and T are sets with cardinality s and t, respectively (C(S)=s and C(T)=t, such that S∩T=∅), then s+t = C(S)+C(T)=defC(S∪T).

The fact that C(S∪T) is well defined and has the unique answer C(S)+C(T) can be established by induction.

A ‘natural’ property of set membership is that the identities of the elements are preserved during set operations. If s∈S and t∈T if, then s∈S∪T and t∈S∪T. For all x∈ S∪T, x∈S or x∈T. What underpins all such reasoning is the assumption that operations on collections are reversible. Any collection may be partitioned into smaller sub-collections, and these can be recombined without any change in composition. The totality of the elements involved is invariant.

The addition of numbers is defined to preserve the cardinalities of sets during the operation of set union, just as set union preserves the identities of the constituent elements during the operation.

C(S)+C(T)=C(S∪T), and C(S∪T)=C(S)+C(T).

From the ‘natural’ properties of set union (which include commutativity and associativity of ∪ and ∩) it is easy to derive the properties of symmetry and associativity for the addition of numbers. With regard to symmetry: If C(S)=s and C(T)=t and S∩T=∅, then s+t=C(S)+C(T). But C(S)+C(T)=C(S∪T), C(S∪T)=C(T∪S), C(T∪S)=C(T)+C(S), C(T)+C(S). Therefore s+t=t+s.

On the basis of these assumptions all collections of objects can be counted. Furthermore, since such collections can be partitioned and recombined without any change in the constituent elements, it follows that the numbers and quantities involved are invariant. These seemingly innocuous assumptions as well as assumptions numbered 1 to 5 in the text provide a foundation for an arithmetic that necessarily conserves number under the usual operations of counting and calculating. Without these assumptions arithmetic as we understand it could not exist. Overall, the conservation of number reflects the conservation of objects and collections that is imposed by our conceptualisation on the world of objects to be counted.

1. In this inquiry I do not ask the interesting psychological question as to why persons might feel uncomfortable with uncertainty and have or feel the need for certainty, or indeed of the place of uncertainty in the human condition. This would take me in another direction, possibly needing psychoanalytic theory, and is beyond the scope of my present inquiry. [↑](#footnote-ref-1)
2. Given even the little that is known about Pythagoreans, with its elevation of number to central elements of the universe and the essence of things, this claim of Aristotle is questionable. [↑](#footnote-ref-2)
3. Empiricism and conventionalism together with other social philosophies of mathematics do not correspond well to this tripartite classification. [↑](#footnote-ref-3)
4. Although Piagetian theory suggests that the normal age for moving to the stage of formal operations is 11 years, empirical evidence shows that some younger children attain this stage and a significant number of teenagers do not attain this stage by the age of 16 years (Shayer et al. 1976-78). [↑](#footnote-ref-4)
5. Learners have to overcome unavoidable ‘epistemological obstacles’ (Bachelard 1938) in the learning of mathematics, when for example, their number world is expanded from the natural numbers (with a smallest number) to the integers (no smallest number). [↑](#footnote-ref-5)
6. In earlier writings (Ernest 1991, 1998) I have used the term certainty to mean absolute certainty, and have rejected the claim that mathematical knowledge is objective and superhuman and can be known with absolute, indubitable and infallible certainty. By redefining objectivity and certainty in cultural and humanistic ways, relativised to and circumscribed by the limits of human and social knowing, I am happy to acknowledge the certainty of mathematical knowledge. [↑](#footnote-ref-6)